university of groningen

Group Theory Final Exam

Date: 02 November 2021

Place: Exam Hall 1, Aletta Jacobs Hall

Time: 15:00 – 18:00

Instructions

- To get full points, you must provide complete arguments and computations. You will get no points if you do not explain your answer. Answers like "True", "False", "34", "Yes", "No" will not be accepted.
- While solving a problem, you can use any statement that needs to be proven as a part of another problem even if you did not manage to prove it; e.g. you can use part (a) while solving part (b) even if you did not prove (a).
- Clearly write your name and student number on each page you submit.
- The examination consists of 5 questions. You can score up to 36 points and you get 4 points for free. This way you will score in total between 4 and 40 points.

Problems

1. Let S_8 be the permutation group on $\{1, 2, \ldots, 7, 8\}$.

   (a) **[2 points]** Compute the inverse of
   $$\sigma := (4\ 5\ 6\ 7)^{2001}(2\ 3)(1\ 2)(6\ 7\ 8)^{35}.$$

   (b) **[2 points]** What is the sign of σ? Is σ an element in the alternating group A_8?

   (c) **[2 points]** Give an example of two non-conjugate elements of S_8 that have the same order.

   (d) **[2 points]** If $g \in S_8$ is an element with the maximal possible order, what is the order of g?

2. Let G be a group. An isomorphism from G to itself is called an *automorphism*. Denote the set of all automorphisms of G by $\mathrm{Aut}(G)$.

   (a) **[2 points]** Find all automorphisms of $\mathbb{Z}/8\mathbb{Z}$.

   (b) **[2 points]** Let G be any group. Show that the map
   $$\begin{aligned} G \times \mathrm{Aut}(G) &\to \mathrm{Aut}(G) \\ (g, \varphi) &\mapsto g \cdot \varphi, \end{aligned}$$
   where $(g \cdot \varphi)(h) = \varphi(g^{-1}hg)$, is an action.

   (c) **[2 points]** Let $S_{\mathrm{Aut}(G)}$ denote the symmetric group of $\mathrm{Aut}(G)$. Show that the map
   $$f : G \to S_{\mathrm{Aut}(G)}$$
   given by $f(g)(\varphi) = g \cdot \varphi$ is a homomorphism.

   (d) **[3 points]** If G is abelian, show that the homomorphism f in (c) is trivial, i.e., $\#f(G) = 1$.

3. **[3 points]** Let G be a group of order $255 = 3 \cdot 5 \cdot 17$. Show that G has a normal subgroup of order either 3 or 5.

4. (a) **[3 points]** List, up to isomorphism, all abelian groups of order $216 = 2^3 3^3$ which have an element of order 9 and no element of order 12.

   (b) **[2 points]** Let H be the subgroup $H \le \mathbb{Z}^3$ generated by $(3, 1, 2)$ and $(0, 1, 8)$. Find the rank and the elementary divisors of $\mathbb{Z}^3/H$.

   (c) **[3 points]** Let $H = (\mathbb{Z}/18\mathbb{Z})/\langle\bar{6}\rangle$ and $K = (\mathbb{Z}/30\mathbb{Z})^\times$ and $P = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Find the elementary divisors of the direct product $H \times K \times P$.

5. **Prove/Disprove.** For each of the following statements, prove the statement if it is true and disprove it if it is false.

   (a) **[2 points]** The only homomorphism from the dihedral group D_5 to $\mathbb{Z}/9\mathbb{Z}$ is the trivial homomorphism.

   (b) **[2 points]** If H is a subgroup of a group G, then the union

   $$\bigcup_{g \in G} gHg^{-1}$$

   is a normal subgroup in G.

   (c) **[2 points]** If a group is simple and abelian then it is cyclic.

   (d) **[2 points]** There is a transitive action of a group of 14 elements on a set of 8 elements.

GOOD LUCK! ☺

$6\mathbb{Z}/18\mathbb{Z}$ $\mathbb{Z}/6\mathbb{Z}$

$\mathbb{Z}/p\mathbb{Z}$